Probability & Statistics (1)

Combinatorial Analysis II

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Outlines

1. Review
2. Multinomial Coefficient
3. The Number of Integer Solutions of Equations
4. [#3] Assignment
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Review

Permutations: $n!$

Combination:

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!\,r!}, r \le n$$

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

The Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Multinomial Coefficient

如果我們今天有n個物品，要拆成r個group，每個group的大小為n_r個物品，且所有group的物品總數會等於原來的總數n 。

因此，對於第一個group的組合有$\binom{n}{n_1}$種，當到第二個group的時候，因為物品總數已經被取走n_1個，所以組合數有$\binom{n-n_1}{n_2}$種。

所以一直到第r個group之後，總共會有

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$$

Multinomial Coefficient

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$$
$$=\frac{n!}{(n-n_1)!\,n_1!}\frac{(n-n_1)!}{(n-n_1-n_2)!\,n_2!}\cdots\frac{(n-n_1-n_2-\cdots-n_{r-1})!}{\boxed{0!}\,n_r!}$$
$$=\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

$0! = 1$

Multinomial Coefficient

- If $n_1 + n_2 + \cdots + n_r = n$, we define $\binom{n}{n_1, n_2, \ldots, n_r}$ by

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

- Thus, $\binom{n}{n_1, n_2, \ldots, n_r}$ represents the number of possible divisions of n distinct object into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$.

Multinomial Coefficient

範例一

- 假設今天台中洲際棒球場要舉辦比賽，目前有13位場務人員要負責巡視場地是否安排妥當，其中有3人負責內野、4人負責左外野、4人負責右外野、2人負責場內。請問有多少種的組合方式，將13人分配到這四組當中。

Multinomial Coefficient

範例二

- 假設你目前還有12學分需要修完才能畢業，但是你還剩下三個學期，因為你還需要做專題跟碩士班入學考需要準備，你這三個學期最多只能分別修2學分、4學分、6學分的課程，請問你有幾種修課組合的方式?

- 如果剩下兩學期(四上&四下)，分別可以修習6學分，那麼有幾種方式?

Multinomial Coefficient

範例三

- 如果今天有12顆白球，要分為兩組各6顆，請問有幾種分組方式?

- 請問範例三與範例二的計算差異，是因為甚麼所造成呢?

Multinomial Coefficient

The Multinomial Theorem (多項式定理)

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1,\ldots,n_r):\\ n_1+\cdots+n_r=n}} \binom{n}{n_1, n_2, \ldots, n_r} {x_1}^{n_1} {x_2}^{n_2} \ldots {x_r}^{n_r}$$

所有非負整數數值向量$(n_1, n_2, \ldots, n_r)$的加總會符合

$n_1 + n_2 + \cdots + n_r = n$。

此時，$\binom{n}{n_1, n_2, \ldots, n_r}$就稱為多項式係數(multinomial coefficient)。

Multinomial Coefficient

範例四 (取自Sheldon Ross. A First of Course in Probability 8th. pp.11)

- ***EXAMPLE 5d***
- In the first round of a knockout tournament involving $n = 2^m$ players, the n players are divided into $n/2$ pairs, with each of these pairs then playing a game. The losers of the games are eliminated while the winners go on to the next round, where the process is repeated until only a single player remains. Suppose we have a knockout tournament of 8 players.
- How many possible outcomes are there for the initial round? (For instance, one outcome is that 1 beats 2, 3 beats 4, 5 beats 6, and 7 beats 8.)

Multinomial Coefficient

Solution

1st Round

1) 8 players into 4 groups = $\binom{8}{2,2,2,2} = \frac{8!}{2^4}$
2) No ordering ➔ $\frac{8!}{2^4 4!}$
3) Winner will come from any two of them ➔ $\frac{8! 2^4}{2^4 4!}$
4) For this round ➔ $\frac{8!}{4!}$

2nd Round

1) 4 players into 2 groups = $\binom{4}{2,2} = \frac{4!}{2^2}$
2) No ordering ➔ $\frac{4!}{2^2 2!}$
3) Winner will come from any two of them ➔ $\frac{4! 2^2}{2^2 2!}$
4) For this round ➔ $\frac{4!}{2!}$

3rd Round

For this round ➔ $\frac{2!}{1!}$

Total possibilities

$$\frac{8!}{4!}\frac{4!}{2!}\frac{2!}{1!} = 8!$$

Multinomial Coefficient

範例五

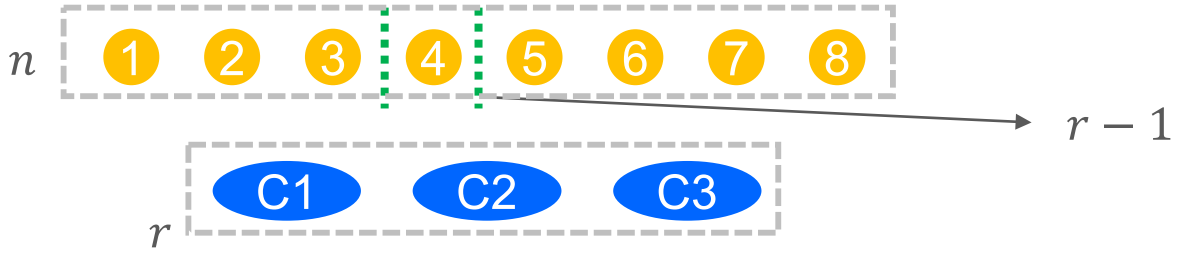
- $(x_1 + x_2 + x_3)^3 = ?$
- $\binom{3}{3,0,0} {x_1}^3 {x_2}^0 {x_3}^0 + \binom{3}{0,3,0} {x_1}^0 {x_2}^3 {x_3}^0 + \binom{3}{0,0,3} {x_1}^0 {x_2}^0 {x_3}^3 + \binom{3}{2,1,0} {x_1}^2 {x_2}^1 {x_3}^0 + \binom{3}{1,2,0} {x_1}^1 {x_2}^2 {x_3}^0 + \binom{3}{0,2,1} {x_1}^0 {x_2}^2 {x_3}^1 + \binom{3}{0,1,2} {x_1}^0 {x_2}^1 {x_3}^2 + \binom{3}{2,0,1} {x_1}^2 {x_2}^0 {x_3}^1 + \binom{3}{1,0,2} {x_1}^1 {x_2}^0 {x_3}^2 + \binom{3}{1,1,1} {x_1}^1 {x_2}^1 {x_3}^1$
- $= {x_1}^3 + {x_2}^3 + {x_3}^3 + 3{x_1}^2 x_2 + 3{x_2}^2 x_1 + 3{x_2}^2 x_3 + 3{x_3}^2 x_2 + 3{x_1}^2 x_3 + 3{x_3}^2 x_1 + 6 x_1 x_2 x_3$

The Number of Integer Solutions of Equations

Proposition 1

- 存在$\binom{n-1}{r-1}$個正整數值向量$(x_1, x_2, \ldots, x_r)$會滿足下列式子
- $x_1 + x_2 + \cdots + x_r = n, \quad x_i > 0, \ i = 1, \ldots, r$

- 如果今天有8顆球(n)要分給3個小孩(r)，且每個人至少要有1顆($x_i > 0, \ i = 1, \ldots, r$)，請問有幾種分配方式?



The Number of Integer Solutions of Equations

- 但如果我們改變情況，小孩可以沒有被分到任何一顆球($x_i \in nonnegative,\ \ i = 1, \dots, r$)，那麼分配的組合有幾種?

- 其實，答案就是剛剛正整數解總數+變數(小孩)數量，因為每一個變數(小孩)都有可能是0；換句話說，也就是$n + r$的意思。

Proposition 2

- 存在$\binom{n+r-1}{r-1}$個非負整數解向量$(x_1, x_2, \dots, x_r)$會滿足下列式子
- $x_1 + x_2 + \cdots + x_r = n$

The Number of Integer Solutions of Equations

範例六

- (1) 請問$x_1 + x_2 + x_3 = 4$存在幾個正整數解?
- Hint: $\binom{n-1}{r-1}$

- (2) 呈上題，有多少個非負整數解?
- Hint: $\binom{n+r-1}{r-1}$

The Number of Integer Solutions of Equations

範例七

- 假設你現在是證券公司股票投資分析師，目前手握10億資金，每次投資單位為1億。根據目前財報基本面分析結果指出，共有6個適合的投資標的。
- (1) 假設所有錢**都**必須進入股票市場，那麼會有多少種投資組合?

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10$$

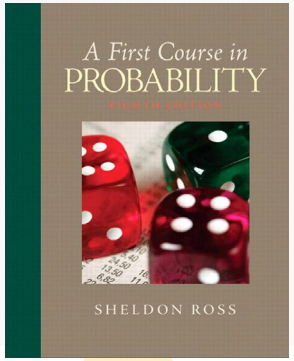
The Number of Integer Solutions of Equations

範例七

- 假設你現在是證券公司股票投資分析師，目前手握10億資金，每次投資單位為1億。根據目前財報基本面分析結果指出，共有6個適合的投資標的。
- (2) 假設所有錢**不**必須進入股票市場，那麼會有多少種投資組合?

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 10$$

[#3] Assignment



- Selected Problems from Sheldon Ross Textbook [1].

3. Twenty workers are to be assigned to 20 different jobs, one to each job. How many different assignments are possible?

4. John, Jim, Jay, and Jack have formed a band consisting of 4 instruments. If each of the boys can play all 4 instruments, how many different arrangements are possible? What if John and Jim can play all 4 instruments, but Jay and Jack can each play only piano and drums?

5. For years, telephone area codes in the United States and Canada consisted of a sequence of three digits. The first digit was an integer between 2 and 9, the second digit was either 0 or 1, and the third digit was any integer from 1 to 9. How many area codes were possible? How many area codes starting with a 4 were possible?

2. If 4 Americans, 3 French people, and 3 British people are to be seated in a row, how many seating arrangements are possible when people of the same nationality must sit next to each other?

3. A president, treasurer, and secretary, all different, are to be chosen from a club consisting of 10 people. How many different choices of officers are possible if

(a) there are no restrictions?

(b) A and B will not serve together?

(c) C and D will serve together or not at all?

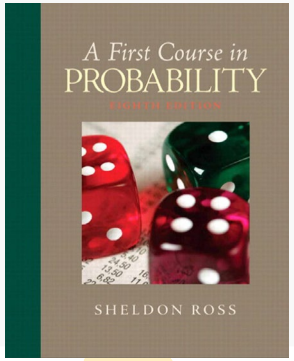
(d) E must be an officer?

(e) F will serve only if he is president?

4. A student is to answer 7 out of 10 questions in an examination. How many choices has she? How many if she must answer at least 3 of the first 5 questions?

[1] Sheldon Ross. A First of Course in Probability. 8th edition.

[#3] Assignment



- Selected Problems from Sheldon Ross Textbook [1].

5. In how many ways can a man divide 7 gifts among his 3 children if the eldest is to receive 3 gifts and the others 2 each?

6. How many different 7-place license plates are possible when 3 of the entries are letters and 4 are digits? Assume that repetition of letters and numbers is allowed and that there is no restriction on where the letters or numbers can be placed.

[1] Sheldon Ross. A First of Course in Probability. 8th edition.

Reference

Ross, S. (2010). *A first course in probability*. Pearson.

Question Time

If you have any questions, please do not hesitate to ask me.

The End

Thank you for your attention))